A note on bipartite graph tiling

Andrzej Czygrinow*, Louis DeBiasio
School of Mathematical and Statistical Sciences
Arizona State University
Tempe, AZ 85287

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Abstract

Bipartite graph tiling was studied by Zhao [7] who gave the best possible minimum degree conditions for a balanced bipartite graph on $2ms$ vertices to contain m vertex disjoint copies of $K_{s,s}$. Let $s < t$ be fixed positive integers. Hladký and Schacht [3] gave minimum degree conditions for a balanced bipartite graph on $2m(s+t)$ vertices to contain m vertex disjoint copies of $K_{s,t}$. Their results were best possible, except in the case when m is odd and $t > 2s+1$. We give the best possible minimum degree condition in this case.

1 Introduction

If G is a graph on $n = sm$ vertices, H is a graph on s vertices and G contains m vertex disjoint copies of H, then we say G can be *tiled* with H. In this language, we state the seminal result of Hajnal and Szemerédi.

Theorem 1.1 (Hajnal-Szemerédi [2]). *Let G be a graph on $n = sm$ vertices. If $\delta(G) \geq (s-1)m$, then G can be tiled with K_s.*

For tiling with general H, results of Alon and Yuster [1] and Komlós, Sárközy, and Szemerédi [4] gave sufficient conditions on the minimum degree of a graph G such that G can be tiled with H. Specifically, in [4], it is shown that if G is a graph on n vertices with minimum degree at least $(1 - 1/\chi(H))\, n + K$ for a constant K that only depends on H, then G can be tiled with H. A more delicate minimum degree condition that involves the so-called critical chromatic number of H was conjectured by Komlós and solved by Shokoufandeh and Zhao [6]. Finally, Kühn and Osthus [5] determined exactly when the critical chromatic number or chromatic number is the appropriate parameter and thus settled the problem (for large graphs).

In this paper we study the tiling problem in bipartite graphs. Denote a bipartite graph G with partition sets U and V by $G[U, V]$. We say $G[U, V]$ is *balanced* if $|U| = |V|$. Zhao proved the following Hajnal-Szemerédi type result for bipartite graphs.

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Theorem 1.2 (Zhao [7]). *For each $s \geq 2$, there exists m_0 such that the following holds for all $m \geq m_0$. If G is a balanced bipartite graph on $2n = 2ms$ vertices with*

$$\delta(G) \geq \begin{cases} \frac{n}{2} + s - 1 & \text{if } m \text{ is even} \\ \frac{n+3s}{2} - 2 & \text{if } m \text{ is odd,} \end{cases}$$

then G can be tiled with $K_{s,s}$.

Zhao proved that this minimum degree condition was tight.

Proposition 1.3 (Zhao [7]). *Let $s \geq 2$, and $n = ms \geq 64s^2$. There exists a balanced bipartite graph, G, on $2n$ vertices with*

$$\delta(G) = \begin{cases} \frac{n}{2} + s - 2 & \text{if } m \text{ is even} \\ \frac{n+3s}{2} - 3 & \text{if } m \text{ is odd} \end{cases}$$

such that G cannot be tiled with $K_{s,s}$.

Hladký and Schacht extended Zhao's result as follows.

Theorem 1.4 (Hladký-Schacht [3]). *Let $1 \leq s < t$ be fixed integers. There exists m_0 such that the following holds for all $m \geq m_0$. If G is a balanced bipartite graph on $2n = 2m(s+t)$ vertices with*

$$\delta(G) \geq \begin{cases} \frac{n}{2} + s - 1 & \text{if } m \text{ is even} \\ \frac{n+t+s}{2} - 1 & \text{if } m \text{ is odd,} \end{cases}$$

then G can be tiled with $K_{s,t}$.

They proved that this minimum degree condition was tight in all cases except when m is odd and $t > 2s+1$. Note that since we are dealing with balanced bipartite graphs, in any tiling of $G[U, V]$ with $K_{s,t}$ there must be an equal number of copies of $K_{s,t}$ with s vertices in U as copies of $K_{s,t}$ with t vertices in U. This explains why the authors [3] suppose $2n = 2m(s+t)$ instead of $2n = m(s+t)$.

Proposition 1.5 (Hladký-Schacht [3]). *Let $1 \leq s < t$ be fixed integers. There exists m_0 such that the following holds for all $m \geq m_0$. There exists a balanced bipartite graph, G, on $2n = 2m(s+t)$ vertices with*

$$\delta(G) = \begin{cases} \frac{n}{2} + s - 2 & \text{if } m \text{ is even} \\ \frac{n+t+s}{2} - 2 & \text{if } m \text{ is odd and } t \leq 2s+1 \end{cases}$$

such that G cannot be tiled with $K_{s,t}$.

Our objective is to give the tight minimum degree condition in the final remaining case, when m is odd and $t > 2s+1$. We will do this in two parts. First in Section 2.3 we prove that when m is odd and $t \geq 2s+1$, the following minimum degree condition is sufficient.

Theorem 1.6. *Let $1 \leq s < t$ be fixed integers with $2s+1 \leq t$. There exists m_0 such that the following holds for all odd m with $m \geq m_0$. If G is a balanced bipartite graph on $2n = 2m(s+t)$ vertices with*

$$\delta(G) \geq \frac{n+3s}{2} - 1,$$

then G can be tiled with $K_{s,t}$.

Then in Section 3 we prove that the minimum degree condition in Theorem 1.6 is tight.

Proposition 1.7. *Let $1 \le s < t$ be fixed integers with $2s+1 \le t$. There exists m_0 such that the following holds for all odd m with $m \ge m_0$. There exists a balanced bipartite graph, G, on $2n = 2m(s+t)$ vertices with*

$$\delta(G) = \begin{cases} \frac{n+3s}{2} - \frac{3}{2} & \text{if } t \text{ is odd} \\ \frac{n+3s}{2} - 2 & \text{if } t \text{ is even} \end{cases}$$

such that G cannot be tiled with $K_{s,t}$.

Let $m = 2k+1$ for some $k \in \mathbb{N}$ and let $n = m(s+t)$. We note that when $t = 2s+1$, $\frac{n+3s}{2} - 1 = (k+1)(s+t) - \frac{3}{2}$ and $\frac{n+t+s}{2} - 1 = (k+1)(s+t) - 1$. So the value for the lower bound in Theorem 1.6 is smaller than the value for the lower bound in Theorem 1.4 when $t = 2s+1$, but since $\delta(G)$ only takes integer values the minimum degree condition in Theorem 1.6 is not an improvement until $t > 2s+1$.

2 Proof of Theorem 1.6

For disjoint sets $A, B \subseteq V(G)$, we define $e(A,B)$ to be the number of edges with one end in A and the other end in B and for $v \in V(G) \setminus A$ we write $\deg(v,A)$ instead of $e(\{v\},A)$. Also, $d(A,B) = \frac{e(A,B)}{|A||B|}$, $\delta(A,B) = \min\{\deg(v,B) : v \in A\}$ and $\Delta(A,B) = \max\{\deg(v,B) : v \in A\}$. An *$h$-star from A to B*, is a copy of $K_{1,h}$ with the vertex of degree h, *the center*, in A and the vertices of degree 1, *the leaves*, in B.

The following theorem appears in [7].

Theorem 2.1 (Zhao [7])**.** *For every $\alpha > 0$ and every positive integer r, there exist $\beta > 0$ and positive integer m_1 such that the following holds for all $n = mr$ with $m \ge m_1$. Given a bipartite graph $G[U,V]$ with $|U| = |V| = n$, if $\delta(G) \ge (\frac{1}{2} - \beta)n$, then either G can be tiled with $K_{r,r}$, or there exist*

$$U_1' \subseteq U,\ V_2' \subseteq V, \quad \text{such that } |U_1'| = |V_2'| = \lfloor n/2 \rfloor,\ d(U_1', V_2') \le \alpha. \tag{1}$$

If a balanced bipartite graph $G[U,V]$ on $2n$ vertices with n divisible by r satisfies (1), we say G is *extremal* with parameter α. In this case we set $U_2' := U \setminus U_1'$ and $V_1' := V \setminus V_2'$.

If we replace r with $s+t$ in Theorem 2.1, we see that either G can be tiled with $K_{s+t,s+t}$ or else we are in the extremal case. If it is the case that G can be tiled with $K_{s+t,s+t}$, we split each copy of $K_{s+t,s+t}$ into two copies of $K_{s,t}$ to give the desired tiling. So we must only deal with the extremal case.

2.1 Pre-processing

Claim 2.2. *Let $0 < \alpha \ll 1$, $r \in \mathbb{N}$ and let $m_1 \in \mathbb{N}$ be given by Theorem 2.1. Let $m \ge m_1$ and suppose that $G[U,V]$ is a balanced bipartite graph on $2n = 2mr$ vertices such that $\delta(G) = \frac{n}{2} + C$, where $0 \le C \le 3r/2$. Suppose further that the deletion of any edge of G will cause the resulting graph to have minimum degree less than $\frac{n}{2} + C$. If G is extremal with parameter α, then $d(U_2', V_1') \le 5\sqrt{\alpha}$.*

Proof. Let $\gamma := 5\sqrt{\alpha}$ and suppose $d(U_2', V_1') > \gamma$. Let $X' = \{u \in U_2' : \deg(u, V_2') < (1-\sqrt{\alpha})\frac{n}{2}\}$, $Y' = \{v \in V_1' : \deg(v, U_1') < (1-\sqrt{\alpha})\frac{n}{2}\}$. Since $e(U_1', V_2') \le \alpha\frac{n^2}{4}$ and $e(U_1', V) \ge |U_1'|\frac{n}{2}$, we have $e(U_1', V_1') \ge |U_1'|\frac{n}{2} - \alpha\frac{n^2}{4}$. Thus we can bound the non-edges between U_1' and V_1',

$$\sqrt{\alpha}\frac{n}{2}|Y'| \le \bar{e}(U_1', V_1') \le \alpha\frac{n^2}{4},$$

which gives $|Y'| \le \sqrt{\alpha}\frac{n}{2}$. Similarly we have $|X'| \le \sqrt{\alpha}\frac{n}{2}$. Let $U_2'' = U_2' \setminus X'$ and $V_1'' = V_1' \setminus Y'$. Since $d(U_2', V_1') > \gamma$, we have

$$e(U_2'', V_1'') \ge \gamma\frac{n^2}{4} - 2\sqrt{\alpha}\frac{n^2}{4} = 3\sqrt{\alpha}\frac{n^2}{4}. \tag{2}$$

Let $X'' = \{u \in U_2'' : \deg(u, V_1'') \ge \sqrt{\alpha}\frac{n}{2}+C+1\}$ and $Y'' = \{v \in V_1'' : \deg(v, U_2'') \ge \sqrt{\alpha}\frac{n}{2}+C+1\}$. If there is an edge $uv \in E(X'', Y'')$, then $\deg(u), \deg(y) \ge \frac{n}{2} + C + 1$ which contradicts the edge minimality of G, so suppose $e(X'', Y'') = 0$. Finally, by (2) we have

$$3\sqrt{\alpha}\frac{n^2}{4} \le e(U_2'', V_1'') \le e(X'', Y'') + e(U_2'' \setminus X'', V_1'') + e(V_1'' \setminus Y'', U_2'') \le 0 + 2(\sqrt{\alpha}\frac{n}{2} + C)\frac{n}{2},$$

which is a contradiction, since n is sufficiently large.

□

Let $1 \le s < t$ be integers so that $2s + 1 \le t$, and let $0 < \alpha \ll 1$ (setting $\alpha := \left(\frac{1}{32t(s+t)}\right)^3$ is small enough). Let $G[U, V]$ be a balanced bipartite graph on $2n = 2m(s + t)$ vertices, where $m = 2k + 1$ and k is a sufficiently large integer with respect to $(\frac{\alpha}{5})^2$. Suppose that G is extremal with parameter $(\frac{\alpha}{5})^2$ and edge-minimal with respect to the condition $\delta(G) \ge \frac{n+3s}{2} - 1$. By Claim 2.2 we have $d(U_i', V_{3-i}') \le \alpha$ for $i = 1, 2$. Then for $i = 1, 2$, we define

$$\begin{aligned}
&U_i = \{u \in U : \deg(u, V_{3-i}') < \alpha^{\frac{1}{3}}\frac{n}{2}\},\ V_i = \{v \in V : \deg(v, U_{3-i}') < \alpha^{\frac{1}{3}}\frac{n}{2}\},\\
&U_0 = U - U_1 - U_2,\ \text{and}\ V_0 = V - V_1 - V_2.
\end{aligned}$$

As a consequence of these definitions, we have the following.

Claim 2.3. *For $i = 1, 2$*

(i) $(1-\alpha^{2/3})\frac{n}{2} \le |U_i|, |V_i| \le (1+\alpha^{2/3})\frac{n}{2}$, (ii) $|U_0|, |V_0| \le \alpha^{2/3}n$,

(iii) $(1-2\alpha^{1/3})\frac{n}{2} < \delta(U_i, V_i), \delta(V_i, U_i)$, (iv) $(\alpha^{1/3}-\alpha^{2/3})\frac{n}{2} \le \delta(U_0, V_i), \delta(V_0, U_i)$,

(v) $\Delta(U_i, V_{3-i}), \Delta(V_{3-i}, U_i) \le \alpha^{1/3}n$

Proof. A proof of (i)-(iv) can be found in [7] and was also used in [3]. So we prove (v) here.

Let $i \in \{1, 2\}$ and note that

$$|U_i' \setminus U_i|\alpha^{1/3}\frac{n}{2} \le e(U_i' \setminus U_i, V_{3-i}') \le e(U_i', V_{3-i}') \le \alpha\frac{n^2}{4} \tag{3}$$

and

$$|V_i' \setminus V_i|\alpha^{1/3}\frac{n}{2} \le e(V_i' \setminus V_i, U_{3-i}') \le e(V_i', U_{3-i}') \le \alpha\frac{n^2}{4}. \tag{4}$$

Then (3) and (4) imply

$$|U_i' \setminus U_i|, |V_i' \setminus V_i| \leq \alpha^{2/3}\frac{n}{2}, \tag{5}$$

which gives $\Delta(U_i, V_{3-i}) \leq \Delta(U_i, V_{3-i}') + |V_{3-i} \setminus V_{3-i}'| \leq \Delta(U_i, V_{3-i}') + |V_i' \setminus V_i| \leq \alpha^{1/3}n$ and $\Delta(V_i, U_{3-i}) \leq \Delta(V_i, U_{3-i}') + |U_{3-i} \setminus U_{3-i}'| \leq \Delta(V_i, U_{3-i}') + |U_i' \setminus U_i| \leq \alpha^{1/3}n$.

□

We need to define some new sets which were not specified in [7].

Definition 2.4. *For $i = 1, 2$, let*

$$\begin{aligned}
&\tilde{U_i} = \{u \in U_i : \deg(u, V_{3-i}) \geq s\},\ \ \tilde{V_i} = \{v \in V_i : \deg(v, U_{3-i}) \geq s\},\\
&\hat{U_i} = U_i \setminus \tilde{U_i},\ \text{and } \hat{V_i} = V_i \setminus \tilde{V_i}.
\end{aligned}$$

Note that the following inequalities are satisfied:

$$\delta(\hat{U_1}, V_0) + \delta(\hat{U_2}, V_0) \geq n + 3s - 2 - (|V_1| + s - 1) - (|V_2| + s - 1) = |V_0| + s \ \text{ and} \tag{6}$$

$$\delta(\hat{V_1}, U_0) + \delta(\hat{V_2}, U_0) \geq n + 3s - 2 - (|U_1| + s - 1) - (|U_2| + s - 1) = |U_0| + s. \tag{7}$$

2.2 Preliminary Claims

The following useful lemma appears in [7].

Lemma 2.5 (Zhao [7], Fact 5.3)**.** *Let $F[A, B]$ be a bipartite graph with $\delta := \delta(A, B)$ and $\Delta := \Delta(B, A)$ Then F contains f_h vertex disjoint h-stars from A to B, and g_h vertex disjoint h-stars from B to A (the stars from A to B and those from B to A need not be disjoint), where*

$$f_h \geq \frac{(\delta - h + 1)|A|}{h\Delta + \delta - h + 1}, \quad g_h \geq \frac{\delta|A| - (h-1)|B|}{\Delta + h\delta - h + 1}.$$

We now prove three claims that we will need in the main proof.

Claim 2.6. *Let $i \in \{1, 2\}$ and $\{A, B\} = \{U_i, V_{3-i}\}$. Let $0 \leq c \leq \alpha^{1/3}n$, $B_0 \subseteq B$ and $A_0 = \{v \in A : \deg(v, B_0) \geq s + c\}$. If $|A_0| \geq \frac{n}{4}$ then there is a set $\mathcal{S}_A$ of at least $\frac{c+1}{8s\alpha^{1/3}}$ vertex disjoint s-stars from A_0 to B_0.*

Proof. Let $\mathcal{S}_A$ be a maximum set of vertex disjoint s-stars from A_0 to B_0 and let $f_s = |\mathcal{S}_A|$. We apply Lemma 2.5 to the graph $G[A_0, B_0]$. Recall, by Claim 2.3, that $\Delta(B, A) \leq \alpha^{1/3}n$. Then

$$f_s \geq \frac{(c+1)|A_0|}{s\alpha^{1/3}n + c + 1} \geq \frac{(c+1)\frac{n}{4}}{2s\alpha^{1/3}n} = \frac{c+1}{8s\alpha^{1/3}}.$$

□

Note that since $n = (2k+1)(s+t)$, we can write $\delta(G) \geq \frac{n+3s}{2} - 1 = k(s+t) + 2s + \frac{t}{2} - 1$.

Claim 2.7. *Let $i \in \{1, 2\}$ and $\{A, B\} = \{U_i, V_{3-i}\}$. Let $|A| = k(s+t) + z$ and $|B| = k(s+t) + y$. Suppose $y \geq z$ and $y \geq \frac{t+1}{2}$. Then there is a set $\mathcal{S}_B$ of y vertex disjoint s-stars with centers $C_B \subseteq B$ and leaves $L_A \subseteq A$. Furthermore if $z \geq 1$, then there is a set $\mathcal{S}_A$ of z vertex disjoint s-stars from $A \setminus L_A$ to $B \setminus C_B$.*

Proof. Let $\beta := 32s\alpha^{1/3}$ and recall that by the choice of α we have $\frac{1}{t} \gg \beta \gg 2\alpha^{1/3}$. We show that the desired set $\mathcal{S}_B$ exists by applying Lemma 2.5 to the graph $G[A,B]$. We have $\delta(A,B) \geq k(s+t)+2s+\frac{t}{2}-1-(n-|B|) = y+s-\frac{t}{2}-1$ and $\Delta(B,A) \leq \alpha^{1/3}n$ by Claim 2.3. Let $g_s = |\mathcal{S}_B|$, then

$$\begin{aligned} g_s &\geq \frac{(y-\frac{t}{2}+s-1)(k(s+t)+z)-(s-1)(k(s+t)+z+y-z)}{\alpha^{1/3}n+s(y-\frac{t}{2}+s-1)-s+1} \\ &= \frac{(y-\frac{t}{2})(k(s+t)+z)-(s-1)(y-z)}{\alpha^{1/3}n+s(y-\frac{t}{2})+s^2-2s+1} \\ &\geq \frac{(y-\frac{t}{2})\frac{n}{3}}{2\alpha^{1/3}n} \qquad \text{(since } y \leq \alpha^{2/3}\frac{n}{2} \text{ and } -\alpha^{2/3}\frac{n}{2} \leq z, \text{ by Claim 2.3)} \\ &\geq y \qquad \text{(since } y \geq \frac{t+1}{2} \text{ and } \alpha \ll 1). \end{aligned}$$

Thus the desired set $\mathcal{S}_B$ exists.

Suppose $z \geq 1$. Let $c := \frac{1}{2}y$ if $y \geq 1/\beta$, and let $c := 0$ if $y < 1/\beta$. Let $B_0 = B \setminus C_B$ and $A_0 = \{v \in A \setminus L_A |\, \deg(v, B_0) \geq s+c\}$ and $\bar{A} = (A \setminus L_A) \setminus A_0$. Suppose that $|\bar{A}| \geq \frac{n}{16}$. Then there exists $u \in C_B$ such that if $y < 1/\beta$,

$$\deg(u,A) \geq \frac{e(\bar{A},C_B)}{|C_B|} \geq \frac{\left(y-\frac{t}{2}+s-1-(s-1)\right)\frac{n}{16}}{y} = \frac{\left(y-\frac{t}{2}\right)\frac{n}{16}}{y} > \frac{\beta n}{32} \geq \alpha^{1/3}n$$

and if $y \geq 1/\beta$,

$$\deg(u,A) \geq \frac{e(\bar{A},C_B)}{|C_B|} > \frac{\left(y-\frac{t}{2}+s-1-(s+\frac{1}{2}y)\right)\frac{n}{16}}{y} = \frac{\left(\frac{y}{2}-\frac{t}{2}-1\right)\frac{n}{16}}{y} > \frac{n}{64} \geq \alpha^{1/3}n,$$

each contradicting Claim 2.3. So $|\bar{A}| < \frac{n}{16}$ and thus $|A_0| \geq |A|-|L_A|-\frac{n}{16} \geq k(s+t)-s\alpha^{2/3}\frac{n}{2}-\frac{n}{16} \geq \frac{n}{4}$. Now let $\mathcal{S}_A$ be a maximum set of disjoint s-stars from A_0 to B_0 and let $f_s = |\mathcal{S}_A|$. By Lemma 2.6 we have $f_s \geq \frac{c+1}{8s\alpha^{1/3}}$. Recall that $1 \leq z \leq y$. If $y \geq 1/\beta$, then $f_s \geq \frac{y}{16s\alpha^{1/3}} \geq z$ and if $y < 1/\beta$, then $f_s \geq \frac{1}{8s\alpha^{1/3}} \geq \frac{1}{\beta} \geq z$. So the desired set $\mathcal{S}_A$ exists.

□

Claim 2.8. *Suppose $|U_0|, |V_0| \geq s$. If $|\hat{U}_1| \geq \frac{n}{8}$ and $|\hat{U}_2| \geq \frac{n}{8}$ (see Definition 2.4), then there is a $K_{s,t} =: K^1$ with s vertices in V_0, $\lceil t/2 \rceil$ vertices in U_1 and $\lfloor t/2 \rfloor$ vertices in U_2. Likewise, if $|\hat{V}_1| \geq \frac{n}{8}$ and $|\hat{V}_2| \geq \frac{n}{8}$ then there is a $K_{s,t} =: K^2$ with s vertices in U_0, $\lceil t/2 \rceil$ vertices in V_1 and $\lfloor t/2 \rfloor$ vertices in V_2.*

Proof. Without loss of generality we will only prove the first statement. Let

$$\ell := s\binom{|U_2|}{\lfloor t/2 \rfloor} \Big/ \binom{\lceil (\alpha^{1/3}-\alpha^{2/3})n/2 \rceil}{\lfloor t/2 \rfloor}$$

and recall that $|U_1|, |U_2| \leq (1+\alpha^{2/3})\frac{n}{2}$ by Claim 2.3. Thus we have

$$\ell \leq s\left(\frac{|U_2|}{(\alpha^{1/3}-\alpha^{2/3})\frac{n}{2}-\lfloor t/2 \rfloor}\right)^{\lfloor t/2 \rfloor} \leq s\left(\frac{(1+\alpha^{2/3})\frac{n}{2}}{(\alpha^{1/3}-\alpha^{2/3})\frac{n}{3}}\right)^{\lfloor t/2 \rfloor} \leq s\left(\frac{3(1+\alpha^{2/3})}{2(\alpha^{1/3}-\alpha^{2/3})}\right)^{\lfloor t/2 \rfloor}. \quad (8)$$

Case 1. $|V_0| \geq \ell\binom{|U_1|}{\lceil t/2\rceil}/\binom{\lceil(\alpha^{1/3}-\alpha^{2/3})n/2\rceil}{\lceil t/2\rceil}$. Recall that $\delta(V_0, U_i) \geq (\alpha^{1/3} - \alpha^{2/3})n/2$ for $i = 1, 2$ by Claim 2.3 and suppose that there is no $K_{\lceil t/2\rceil,\ell}$ with $\lceil t/2\rceil$ vertices in U_1 and ℓ vertices in V_0. We count the $\lceil t/2\rceil$-stars from V_0 to U_1 in two ways which gives

$$|V_0|\binom{\lceil(\alpha^{1/3} - \alpha^{2/3})n/2\rceil}{\lceil t/2\rceil} < \ell\binom{|U_1|}{\lceil t/2\rceil}$$

contradicting the lower bound for $|V_0|$. Consequently there is a complete bipartite graph $K' = K_{\lceil t/2\rceil,\ell}$ with $\lceil t/2\rceil$ vertices in U_1 and ℓ vertices in V_0. If there is no $K_{\lfloor t/2\rfloor,s}$ with s vertices in $V(K') \cap V_0$ and $\lfloor t/2\rfloor$ vertices in U_2, then a similar counting argument gives

$$\ell\binom{\lceil(\alpha^{1/3} - \alpha^{2/3})n/2\rceil}{\lfloor t/2\rfloor} < s\binom{|U_2|}{\lfloor t/2\rfloor}$$

contradicting the definition of ℓ.

Case 2. $|V_0| < \ell\binom{|U_1|}{\lceil t/2\rceil}/\binom{\lceil(\alpha^{1/3}-\alpha^{2/3})n/2\rceil}{\lceil t/2\rceil}$. By (8), we have

$$|V_0| < \ell\left(\frac{3(1+\alpha^{2/3})}{2(\alpha^{1/3}-\alpha^{2/3})}\right)^{\lceil t/2\rceil} \leq s\left(\frac{3(1+\alpha^{2/3})}{2(\alpha^{1/3}-\alpha^{2/3})}\right)^{t}.$$

Let $p := \delta(\hat{U}_1, V_0)$, and note that $p \geq s$ by (6). We claim that there is a complete bipartite graph $K' := K_{\lceil t/2\rceil,p}$ with $\lceil t/2\rceil$ vertices in $\hat{U}_1$ and p vertices in V_0. Let c be the number of p-stars with centers in $\hat{U}_1$ and leaves in V_0. We have $c \geq |\hat{U}_1| \geq \frac{n}{8}$ and if no p-subset of V_0 is in $\lceil t/2\rceil$ of such stars, i.e. K' does not exist, we have $c \leq (\lceil t/2\rceil - 1)\binom{|V_0|}{p}$ which contradicts the fact that $|V_0|$ is $O(1)$ and n is sufficiently large (with respect to α, t, and consequently $|V_0|$). From (6) we have $\delta(\hat{U}_2, V_0) \geq |V_0| - p + s$, so every vertex $u \in \hat{U}_2$ has at least s neighbors in $V(K') \cap V_0$. Repeating the argument above by counting s-stars with centers in $\hat{U}_2$ and leaves in $V(K') \cap V_0$ gives $K'' := K_{s,\lfloor t/2\rfloor}$. Now choose $K^1 \subseteq K' \cup K''$ having the property that $|V_0 \cap V(K^1)| = s$, $|U_1 \cap V(K^1)| = \lceil t/2\rceil$, and $|U_2 \cap V(K^1)| = \lfloor t/2\rfloor$ as desired. $\square$

2.3 Extremal Case

Recall that $t \geq 2s + 1$, $n = (2k + 1)(s + t)$ for some sufficiently large $k \in \mathbb{N}$, and $\delta(G) \geq \frac{n+3s}{2} - 1 = k(s + t) + 2s + \frac{t}{2} - 1$. We start with the partition given in Section 2.1 and we call U_0 and V_0 the *exceptional* sets. Let $i \in \{1, 2\}$. We will attempt to update the partition by moving a constant number (depending only on t) of *special* vertices between U_1 and U_2, denote them by X, and *special* vertices between V_1 and V_2, denote them by Y, as well as partitioning the exceptional sets as $U_0 = U_0^1 \cup U_0^2$ and $V_0 = V_0^1 \cup V_0^2$. Let U_1^*, U_2^*, V_1^* and V_2^* be the resulting sets after moving the special vertices. Our goal is to obtain two graphs, $G_1 := G[U_1^* \cup U_0^1, V_1^* \cup V_0^1]$ and $G_2 := [U_2^* \cup U_0^2, V_2^* \cup V_0^2]$ so that G_1 satisfies

$$|U_1^* \cup U_0^1| = \ell_1(s + t) + as + bt, |V_1^* \cup V_0^1| = \ell_1(s + t) + bs + at$$

and G_2 satisfies

$$|U_2^* \cup U_0^2| = \ell_2(s + t) + bs + at, |V_2^* \cup V_0^2| = \ell_2(s + t) + as + bt,$$

for some nonnegative integers a, b, ℓ_1, ℓ_2. We tile G_1 as follows. We find a copies of $K_{s,t}$, each with t vertices in U_1^*, so that each special vertex in $X \cap U_1^*$ is in a unique copy (some copies may not contain any special vertex). Also, we find b copies of $K_{s,t}$, each with t vertices in V_1^* so that each special vertex in $Y \cap V_1^*$ is in a unique copy (some copies may not contain any special vertex). Note that we only move vertices which will make this step possible. Deleting these $a+b$ copies of $K_{s,t}$ from G_1 gives us a balanced bipartite graph on $2\ell_1(s+t)$ vertices. As noted in [7] and [3], this graph can easily be tiled: By Claim 2.3 there are at most $\alpha^{2/3}\frac{n}{2}$ exceptional vertices in U_0^1 (resp. V_0^1), each with degree at least $(\alpha^{1/3} - \alpha^{2/3})\frac{n}{2}$ to V_1 (resp. U_1), so they may greedily be incorporated into unique copies of $K_{s+t,s+t}$. The remaining graph is still balanced, divisible by $s+t$, and almost complete, thus can be tiled.

So if we are able to split G into graphs G_1 and G_2 as detailed above, we will conclude that G can be tiled. However, if it is not possible to carry out this goal, then we will use an alternate method which is explained in Case 2.

Proof of Theorem 1.6. There are two main cases.
Case 1. $\max\{|U_1|, |U_2|, |V_1|, |V_2|\} \geq k(s+t) + \frac{t+1}{2}$. Without loss of generality, suppose $|U_1| = \max\{|U_1|, |U_2|, |V_1|, |V_2|\}$.

Case 1.1. $|V_2 \cup V_0| \geq k(s+t)+s$. We apply Claim 2.7 to $G[U_1, V_2]$ with $A = V_2$ and $B = U_1$ to obtain $|U_1| - (k(s+t)+s)$ vertex disjoint s-stars with centers $C_U \subseteq U_1$ and leaves in V_2 and a set of $\max\{0, |V_2| - (k(s+t)+s)\}$ vertex disjoint s-stars with centers $C_V \subseteq V_2$ and leaves in U_1. We move the vertices in C_U to U_2 and the vertices in C_V to V_1. If $|V_2| < k(s+t)+s$, we choose $V_0' \subseteq V_0$ so that $|(V_2 \cup V_0) \setminus V_0')| = k(s+t)+s$ otherwise we set $V_0' = \emptyset$. Then $G_1 := G[U_1 \setminus C_U, V_1 \cup C_V \cup V_0']$ satisfies

$$|U_1| - |C_U| = k(s+t)+s, |V_1| + |V_0'| + |C_V| = k(s+t)+t,$$

and $G_2 := G - G_1$ satisfies

$$|U_2 \cup U_0| + |C_U| = k(s+t)+t, |V_2| + |V_0 \setminus V_0'| - |C_V| = k(s+t)+s.$$

Thus G_1 and G_2 can be tiled, which completes the tiling of G.

Case 1.2. $|V_2 \cup V_0| < k(s+t)+s$.

This implies $|V_1| > k(s+t)+t$. So we apply Claim 2.7 to $G[V_1, U_2]$ with $A = U_2$ and $B = V_1$ to obtain a set of $|V_1| - k(s+t)$ vertex disjoint s-stars with centers $C_V \subseteq V_1$ and leaves in U_2. Likewise we apply Claim 2.7 to $G[U_1, V_2]$ with $A = V_2$ and $B = U_1$ to obtain a set of $|U_1| - k(s+t)$ vertex s-stars with centers $C_U \subseteq U_1$ and leaves in V_2. We move the vertices in C_U to U_2 and the vertices in C_V to V_2. Then $G_1 := G[U_1 \setminus C_U, V_1 \setminus C_V]$ satisfies

$$|U_1| - |C_U| = k(s+t), |V_1| - |C_V| = k(s+t)$$

and $G_2 := G - G_1$ satisfies

$$|U_2 \cup U_0| + |C_U| = (k+1)(s+t), |V_2 \cup V_0| + |C_V| = (k+1)(s+t).$$

Thus G_1 and G_2 can be tiled, which completes the tiling of G.
Case 2. $\max\{|U_1|, |U_2|, |V_1|, |V_2|\} \leq k(s+t) + \frac{t}{2}$. Note that this implies $|U_0|, |V_0| \geq s$.

Case 2.1. $\max\{|\tilde{U}_1|, |\tilde{U}_2|, |\tilde{V}_1|, |\tilde{V}_2|\} \geq \frac{n}{4}$ (see Definition 2.4). Without loss of generality we can assume $|\tilde{U}_1| = \max\{|\tilde{U}_1|, |\tilde{U}_2|, |\tilde{V}_1|, |\tilde{V}_2|\}$. Set $h := \lceil t/(2s) \rceil$. Since $|\tilde{U}_1| > \frac{n}{4}$ and $\frac{1}{8s\alpha^{1/3}} \geq (h-1)(s+t)$, we can apply Claim 2.6 to $G[\tilde{U}_1, V_2]$ with $c = 0$ to obtain a set of $(h-1)(s+t)$ vertex

disjoint s-stars with centers $C_U \subseteq \tilde{U}_1$ and leaves in V_2. We first move the vertices in C_U from $\tilde{U}_1$ to U_2. Then since

$$\frac{t}{2} = s\frac{t}{2s} \leq sh \leq s\frac{t+2s-1}{2s} = \frac{t}{2} + s - \frac{1}{2},$$

we can choose sets $U_0' \subseteq U_0$ with $|U_0'| = k(s+t) + \lfloor t/2 \rfloor - |U_1| + sh - \lfloor t/2 \rfloor$ and $V_0' \subseteq V_0$ with $|V_0'| = k(s+t) + \lfloor t/2 \rfloor - |V_1| + s + \lceil t/2 \rceil - sh$ so that $G_1 := G[(U_1 \cup U_0') \setminus C_U, V_1 \cup V_0']$ satisfies

$$|U_1| + |U_0'| - |C_U| = (k-h+1)(s+t) + hs, |V_1| + |V_0'| = (k-h+1)(s+t) + ht,$$

and $G_2 := G - G_1$ satisfies

$$|U_2| + |U_0 \setminus U_0'| + |C_U| = k(s+t) + ht, |V_2| + |V_0 \setminus V_0'| = k(s+t) + hs.$$

Thus G_1 and G_2 can be tiled, which completes the tiling of G.

Case 2.2. $\max\{|\tilde{U}_1|, |\tilde{U}_2|, |\tilde{V}_1|, |\tilde{V}_2|\} < \frac{n}{4}$. Thus for $i = 1, 2$, we have

$$|\hat{U}_i|, |\hat{V}_i| \geq (1 - \alpha^{2/3})\frac{n}{2} - \frac{n}{4} \geq \frac{n}{8}.$$

So we may apply Claim 2.8 to obtain the two special copies of $K_{s,t}$, K^1 and K^2. Note that $|U_i \setminus V(K^1)|$, $|V_i \setminus V(K^2)| \leq k(s+t)$ for $i = 1, 2$. Let $U_0' = U_0 \setminus V(K^2)$ and $V_0' = V_0 \setminus V(K^1)$. We remove the graphs K^1 and K^2, then we partition the vertices $U_0' = U_0^1 \cup U_0^2$ and $V_0' = V_0^1 \cup V_0^2$ so that $G_1 := G[(U_1 \cup U_0^1) \setminus V(K^1), (V_1 \cup V_0^1) \setminus V(K^2)]$ satisfies

$$|U_1| - \lceil t/2 \rceil + |U_0^1| = k(s+t), |V_1| - \lceil t/2 \rceil + |V_0^1| = k(s+t)$$

and $G_2 = G - G_1 - K^1 - K^2$ satisfies

$$|U_2| - \lfloor t/2 \rfloor + |U_0^2| = k(s+t), |V_2| - \lfloor t/2 \rfloor + |V_0^2| = k(s+t).$$

Thus G_1 and G_2 can be tiled, so along with K^1 and K^2, this completes the tiling of G.

$\square$

3 Tightness

In this section we will prove Proposition 1.7. We will need to use the graphs $P(m, p)$, where $m, p \in \mathbb{N}$, introduced by Zhao in [7].

Lemma 3.1. *For all $p \in \mathbb{N}$ there exists m_0 such that for all $m \in \mathbb{N}$, $m > m_0$, there exists a balanced bipartite graph, $P(m, p)$, on $2m$ vertices, so that the following hold:*

(i) *$P(m, p)$ is p-regular*

(ii) *$P(m, p)$ does not contain a copy of $K_{2,2}$.*

Proof of Proposition 1.7. Let $G[U, V]$ be a balanced bipartite graph on $2n$ vertices satisfying the following conditions. Let $n = (2k+1)(s+t)$ for some sufficiently large k (as determined by Lemma 3.1 with $p = s-1$). Partition U into $U = U_0 \cup U_1 \cup U_2$ and partition V into $V = V_0 \cup V_1 \cup V_2$ where, $|U_1| = |V_2| = k(s+t) + \left\lfloor \frac{t+1}{2} \right\rfloor$, $|V_1| = |U_2| = k(s+t) + \left\lceil \frac{t+1}{2} \right\rceil$ and $|U_0| = |V_0| = s-1$. Let $G[U_i, V_i]$ be complete for $i \in \{1, 2\}$, $G[U_1, V_2] \cong P\left(k(s+t) + \left\lfloor \frac{t+1}{2} \right\rfloor, s-1\right)$ and $G[U_2, V_1] \cong$

$P\left(k(s+t)+\left\lceil\frac{t+1}{2}\right\rceil, s-1\right)$. Let $G[U_0, V_1 \cup V_2]$ be complete, $G[V_0, U_1 \cup U_2]$ be complete and $G[U_0, V_0]$ be empty. Note that

$$\delta(G) = \begin{cases} \frac{n+3s}{2} - \frac{3}{2} & \text{if } t \text{ is odd} \\ \frac{n+3s}{2} - 2 & \text{if } t \text{ is even.} \end{cases}$$

Finally we reiterate the following properties of $G[U_1, V_2]$ and $G[U_2, V_1]$. For $i = 1, 2$,

$$\Delta(U_i, V_{3-i}) = \Delta(V_i, U_{3-i}) = s - 1 \tag{9}$$

and

$$G[U_i, V_{3-i}] \text{ is } K_{2,2}\text{-free.} \tag{10}$$

For $i \in \{1, 2\}$ and $A \in \{U_i, V_i\}$, let $A^D := V_{3-i}$ if $A = U_i$ and let $A^D := U_{3-i}$ if $A = V_i$. We call A^D the *diagonal set of* A. Let $A^N := V_i$ if $A = U_i$ and $A^N := U_i$ if $A = V_i$. We call A^N the *non-diagonal set of* A. Finally, we let $A^M := V_0$ if $A = U_i$ and $A^M := U_0$ if $A = V_i$. We call A^M the *opposite middle set of* A.

Suppose $K \cong K_{s,t}$ is a subgraph of G. We say K is a *crossing* $K_{s,t}$ if $V(K) \cap (U_1 \cup V_1) \neq \emptyset$ and $V(K) \cap (U_2 \cup V_2) \neq \emptyset$. Let $\mathcal{W} = \{U_1, U_2, V_1, V_2\}$.

Claim 3.2. *If K is a crossing $K_{s,t}$, then*

(i) *$V(K)$ must intersect some member of $\mathcal{W}$ in exactly one vertex, and*

(ii) *there is a unique $A_0 \in \{U_0, V_0\}$ such that $V(K) \cap A_0 \neq \emptyset$.*

Furthermore, if $|V(K) \cap A| = 1$ for some $A \in \mathcal{W}$, then

(iii) *$V(K) \cap A^D \neq \emptyset$, and*

(iv) *either $|V(K) \cap A^N| \geq 2$ and $V(K) \cap (A^N)^D = \emptyset$, or $V(K) \cap A^N = \emptyset$ and $|V(K) \cap (A^N)^D| \geq 2$.*

Proof. *(i)* Suppose not. Then without loss of generality, suppose that $|V(K) \cap V_1| \geq 2$. By (10) we have, $|V(K) \cap U_2| \leq 1$ and thus $V(K) \cap U_2 = \emptyset$. Since K is crossing, we have $V(K) \cap V_2 \neq \emptyset$ and thus $|V(K) \cap V_2| \geq 2$. By (10) we have, $|V(K) \cap U_1| \leq 1$ and thus $V(K) \cap U_1 = \emptyset$. This is a contradiction, since $K \cong K_{s,t}$ and $|V(K) \cap U| \leq |U_0| = s - 1$.

(ii) Suppose first that $V(K) \cap U_0 = \emptyset = V(K) \cap V_0$. By Claim 3.2 (i), we can assume without loss of generality that $|V(K) \cap U_1| = 1$. Then either $|V(K) \cap U_2| = t - 1$ or $|V(K) \cap U_2| = s - 1$. If $|V(K) \cap U_2| = t - 1$, then by (9) we must have $V(K) \cap V_1 = \emptyset$ which implies $|V(K) \cap V_2| = s$, contradicting (9). If $|V(K) \cap U_2| = s - 1$, then since $t \geq 2s + 1$ we have $|V(K) \cap V_1| \geq s + 1$ or $|V(K) \cap V_2| \geq s + 1$, both of which contradict (9). Thus there exists $A_0 \in \{U_0, V_0\}$ such that $V(K) \cap A_0 \neq \emptyset$. Finally since $G[U_0, V_0]$ is empty, A_0 must be unique.

(iii) Suppose that $V(K) \cap A^D = \emptyset$. Since $|V_0| = s - 1$, we have $V(K) \cap A^N \neq \emptyset$ and since K is crossing, we have $V(K) \cap (A^N)^D \neq \emptyset$. Then by (9), we have $|V(K) \cap A^N|, |V(K) \cap (A^N)^D| \leq s - 1$. Thus $|V(K) \cap U| \leq 2s - 1$ and $|V(K) \cap V| \leq 2s - 2$, contradicting the fact that $K \cong K_{s,t}$ and $t \geq 2s + 1$.

(iv) We first show that it is not possible for either $|V(K) \cap A^N| = 1$ or $|V(K) \cap (A^N)^D| = 1$. If $|V(K) \cap A^N| = 1$, then by (9) and $|U_0| = |V_0| = s - 1$, we have $|V(K) \cap U|, |V(K) \cap V| \leq 2s - 1$, contradicting the fact that $K \cong K_{s,t}$ and $t \geq 2s + 1$. So suppose $|V(K) \cap (A^N)^D| = 1$. If

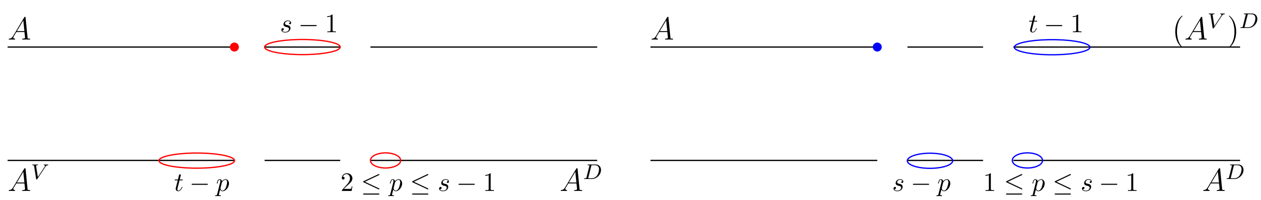
$V(K) \cap U_0 = \emptyset$, then $|V(K) \cap U| = 2$ and since $t \geq 3$ we must have $s = 2$. Then by (9) we have $|V(K) \cap V| \leq 3$ contradicting the fact that $K \cong K_{s,t}$ and $t \geq 2s+1$. If $V(K) \cap U_0 \neq \emptyset$, then $V(K) \cap V_0 = \emptyset$. So $|V(K) \cap U| \leq s+1$ and by (9), $|V(K) \cap V| \leq 2s-2$ contradicting the fact that $K \cong K_{s,t}$ and $t \geq 2s+1$.

Now suppose $V(K) \cap A^N \neq \emptyset$ and $V(K) \cap (A^N)^D \neq \emptyset$. Thus, by the previous paragraph we have $|V(K) \cap A^N|, |V(K) \cap (A^N)^D| \geq 2$, contradicting (10).

So suppose that $V(K) \cap A^N = \emptyset = V(K) \cap (A^N)^D$. Then it must be the case that $|V(K) \cap (A^N)^M| = s-1$ and consequently $|V(K) \cap A^D| = t$, contradicting (9).

□

Let $A \in \mathcal{W}$. We say K is *crossing from* A if either $|V(K) \cap A| = 1$ and $|V(K) \cap A^D| \geq 2$, or $|V(K) \cap A| = 1$, $|V(K) \cap A^D| = 1$ and $V(K) \cap A^M \neq \emptyset$. We say that a crossing $K_{s,t}$ from A is *Type 1* if $|V(K) \cap (A^N)^M| = s-1$, $|V(K) \cap A^N| = t-p$ and $|V(K) \cap A^D| = p$ for some $2 \leq p \leq s-1$. We say that a crossing $K_{s,t}$ from A is *Type 2* if $|V(K) \cap (A^N)^D| = t-1$, $|V(K) \cap A^M| = s-p$, and $|V(K) \cap A^D| = p$ for some $1 \leq p \leq s-1$.



(Type 1 crossing $K_{s,t}$ from A) (Type 2 crossing $K_{s,t}$ from A)

Figure 1

Claim 3.3. *Every crossing $K_{s,t}$ is either Type 1 or Type 2.*

Proof. (See Figure 1) Let K be a crossing $K_{s,t}$ and without loss of generality suppose K is crossing from U_1. Let $p := |V(K) \cap V_2|$. By Claim 3.2 (iii) and (9) we have $1 \leq p \leq s-1$. Suppose K is not Type 1. If $V(K) \cap U_2 = \emptyset$, then $|V(K) \cap U_0| = s-1$ which implies $V(K) \cap V_0 = \emptyset$ by Claim 3.2 (ii). Since K is not Type 1, it must be the case that $|V(K) \cap V_2| = 1$ and $|V(K) \cap V_1| = t-1$ in which case K is not crossing from U_1, contradicting our assumption. So we suppose that $V(K) \cap U_2 \neq \emptyset$. By Claim 3.2 (iv) we have $|V(K) \cap U_2| \geq 2$ and $V(K) \cap V_1 = \emptyset$, which implies that $|V(K) \cap V_0| = s-p$. So by Claim 3.2 (ii), we have $V(K) \cap U_0 = \emptyset$ and thus $|V(K) \cap U_2| = t-1$, so K is Type 2.

□

Suppose for a contradiction that G can be tiled with $K_{s,t}$. Let $\mathcal{F}$ be a tiling of G which minimizes the number of crossing $K_{s,t}$'s.

Claim 3.4. *For $i = 1, 2$, if there is a crossing $K_{s,t}$ of Type 2 from U_i or V_i, then there is no crossing $K_{s,t}$ of Type 2 from U_{3-i} or V_{3-i}.*

Proof. Without loss of generality suppose K^1 is a crossing $K_{s,t}$ of Type 2 from U_1. Suppose that K^2 is a crossing $K_{s,t}$ of Type 2 from U_2 (See Figure 2). For $i \in \{1, 2\}$, let

$$K_*^i := G[U_i \cap (V(K^1) \cup V(K^2)), V(K^{3-i}) \cap (V_0 \cup V_i)].$$

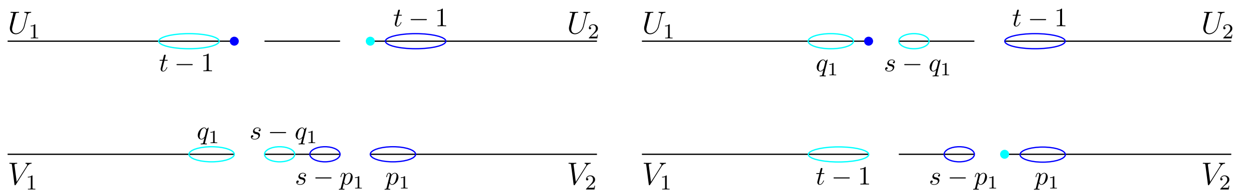


Figure 2: Two cases in the proof of Claim 3.4.

We have $K_*^1 \cong K_{s,t} \cong K_*^2$, neither of K_*^1, K_*^2 are crossing, and $V(K^1) \cup V(K^2) = V(K_*^1) \cup V(K_*^2)$. Thus we obtain a tiling with fewer crossing $K_{s,t}$'s, contradicting the minimality of $\mathcal{F}$.

Now, suppose K^1 is a crossing $K_{s,t}$ of Type 2 from U_1 and K^2 is a crossing $K_{s,t}$ of Type 2 from V_2 (See Figure 2). Specify an element $L^1 \in \mathcal{F}$, such that $V(L^1) \subseteq U_1 \cup V_1$ and $|V(L^1) \cap V_1| = t$ and specify an element $L^2 \in \mathcal{F}$, such that $V(L^2) \subseteq U_2 \cup V_2$ and $|V(L^2) \cap U_2| = t$. Choose arbitrary vertices $v' \in V(K^1) \cap V_0$ and $u' \in V(K^2) \cap U_0$. We now define four subgraphs of G. Let

$$\begin{aligned}
K_*^1 &:= G[V(L^1) \cap V_1, (V(K^1) \cup V(K^2)) \cap ((U_1 \cup U_0) \setminus \{u'\})], \\
L_*^1 &:= G[V(L^1) \cap U_1, (V(K^2) \cap V_1) \cup \{v'\}], \\
K_*^2 &:= G[V(L^2) \cap U_2, (V(K^1) \cup V(K^2)) \cap ((V_2 \cup V_0) \setminus \{v'\})], \text{ and} \\
L_*^2 &:= G[V(L^2) \cap V_1, (V(K^1) \cap U_2) \cup \{u'\}].
\end{aligned}$$

All of $K_*^1, K_*^2, L_*^1, L_*^2$ are isomorphic to $K_{s,t}$, none of $K_*^1, K_*^2, L_*^1, L_*^2$ are crossing, and $V(K_*^1) \cup V(K_*^2) \cup V(L_*^1) \cup V(L_*^2) = V(K^1) \cup V(K^2) \cup V(L^1) \cup V(L^2)$. Thus we obtain a tiling with fewer crossing $K_{s,t}$'s, contradicting the minimality of $\mathcal{F}$. $\square$

For $i \in \{1, 2\}$, let $\mathcal{F}_i$ be the set of all copies of $K_{s,t}$ in $\mathcal{F}$ which touch $U_i \cup V_i$. And let U_i^* (resp. V_i^*) be all the vertices in U (resp. V) which touch elements of $\mathcal{F}_i$. Precisely, let $\mathcal{F}_i = \{K \in \mathcal{F} : V(K) \cap (U_i \cup V_i) \neq \emptyset\}$ for $i = 1, 2$, and let

$$U_i^* = (\cup_{K \in \mathcal{F}_i} V(K)) \cap U \quad \text{and} \quad V_i^* = (\cup_{K \in \mathcal{F}_i} V(K)) \cap V.$$

Note that $U_i \subseteq U_i^*$ and $V_i \subseteq V_i^*$. We will use the following claim to show that all of the remaining possible configurations of crossing $K_{s,t}$'s lead to contradictions.

Claim 3.5. *For all $i \in \{1, 2\}$, either*

$$\max\{|U_i^*|, |V_i^*|\} \geq k(s+t) + 2t \quad \text{or} \quad \min\{|U_i^*|, |V_i^*|\} \geq (k+1)(s+t).$$

Proof. Suppose that $\max\{|U_i^*|, |V_i^*|\} < k(s+t) + 2t$. Then since $U_i \subseteq U_i^*$ and $V_i \subseteq V_i^*$, we have

$$k(s+t) + s < |U_i^*|, |V_i^*| < k(s+t) + 2t, \tag{11}$$

and thus

$$||U_i^*| - |V_i^*|| < 2t - s. \tag{12}$$

By definition $G[U_i^*, V_i^*]$ can be tiled, thus there exists nonnegative integers ℓ, a, b such that $|U_i^*| = \ell(s+t) + as + bt$ and $|V_i^*| = \ell(s+t) + at + bs$. By choosing ℓ to be maximal, we have $a = 0$ or $b = 0$. If $\ell \leq k-1$, then in order to satisfy the lower bound in (11) we must have $a \geq 3$ or $b \geq 3$. Since $a = 0$ or $b = 0$, we have $||U_i^*| - |V_i^*|| \geq 3t - 3s \geq 2t - s$, which contradicts (12). If $\ell = k$, then in order to satisfy the lower bound in (11), we must have $a \geq 2$ or $b \geq 2$, but then we violate the upper bound. So $\ell \geq k+1$ and we have $\min\{|U_i^*|, |V_i^*|\} \geq (k+1)(s+t)$. $\square$

We will also use the following facts. For $i = 1, 2$, we have

$$|V_i \cup V_0| + s, |U_i \cup U_0| + s \le k(s+t) + \frac{t+2}{2} + 2s - 1 < (k+1)(s+t). \tag{13}$$

which in particular implies

$$|V_i \cup V_0| + t, |U_i \cup U_0| + t < k(s+t) + 2t. \tag{14}$$

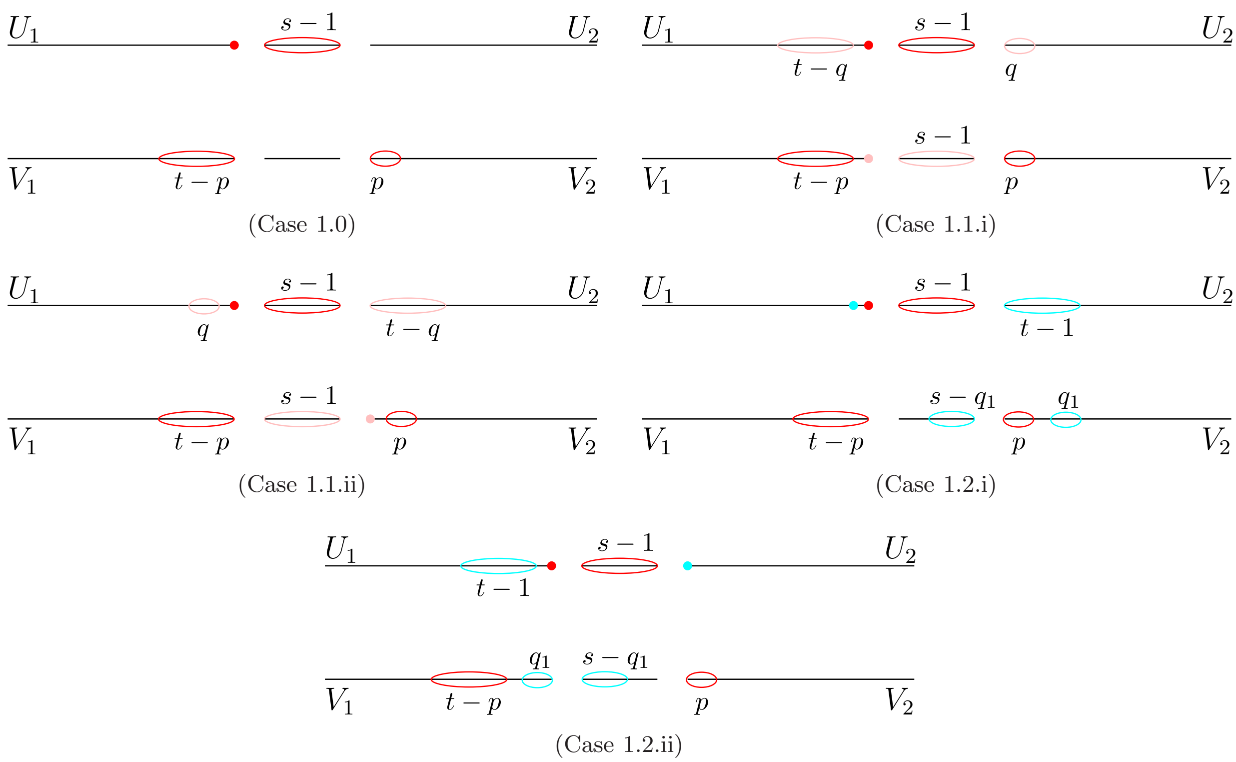


Figure 3: Case 1

Let $i \in \{1, 2\}$ and let $X_i = \{K \in \mathcal{F} : K$ is crossing from U_i and K is Type 2$\}$ and $Y_i = \{K \in \mathcal{F} : K$ is crossing from V_i and K is Type 2$\}$. Since $|U_0| = |V_0| = s - 1$, Claim 3.2 (ii) implies,

$$0 \le |X_i|, |Y_i| \le s - 1. \tag{15}$$

Case 0. There are no crossing $K_{s,t}$'s. So $|U_1^*| \le |U_1 \cup U_0|$ and $|V_1^*| \le |V_1 \cup V_0|$. Then by (13) we have $|U_1^*|, |V_1^*| < (k+1)(s+t)$, contradicting Claim 3.5.

Case 1. There is a crossing $K_{s,t}$ of Type 1. Without loss of generality, suppose K^1 is a crossing $K_{s,t}$ of Type 1 from U_1 and let $p := |V(K^1) \cap V_2|$. Since $U_0 \setminus V(K^1) = \emptyset$, there can be no other crossing $K_{s,t}$'s of Type 1 from U_1 or U_2 and no crossing $K_{s,t}$'s of Type 2 from V_1 or V_2. By Claim 3.3, we must only consider five subcases:

Case 1.0. K^1 is the only crossing $K_{s,t}$. So $|U_1^*| \le |U_1 \cup U_0|$ and $|V_1^*| \le |V_1 \cup V_0| + p < |V_1 \cup V_0| + s$. Then by (13) we have $|U_1^*|, |V_1^*| < (k+1)(s+t)$, contradicting Claim 3.5.

Case 1.1.i. There is a crossing $K_{s,t}$ of Type 1 from V_1. Let K^2 be a crossing $K_{s,t}$ from V_1 and let $q := |V(K^2) \cap U_2|$. Since $V_0 \setminus V(K^2) = \emptyset$, K^1 and K^2 are the only crossing $K_{s,t}$'s. So

$|U_1^*| \leq |U_1 \cup U_0| + q < |U_1 \cup U_0| + s$ and $|V_1^*| \leq |V_1 \cup V_0| + p < |V_1 \cup V_0| + s$. Then by (13) we have, $|U_1^*|, |V_1^*| < (k+1)(s+t)$, contradicting Claim 3.5.

Case 1.1.ii. There is a crossing $K_{s,t}$ of Type 1 from V_2. Let K^2 be a crossing $K_{s,t}$ from V_2 and let $q := |V(K^2) \cap U_1|$. Since $V_0 \setminus V(K^2) = \emptyset$, K^1 and K^2 are the only crossing $K_{s,t}$'s. So $|V_1^*| \leq |V_1 \cup V_0| + p + 1 \leq |V_1 \cup V_0| + s$ and $|U_1^*| \leq |U_1 \cup U_0| + t - q < |U_1 \cup U_0| + t$. Then by (13) and (14) we have $|V_1^*| < (k+1)(s+t)$ and $|U_1^*| < k(s+t) + 2t$, contradicting Claim 3.5.

Case 1.2.i. $1 \leq |X_1|$. By Claim 3.4, since there exists a crossing $K_{s,t}$ of Type 2 from U_1, there can be no crossing $K_{s,t}$'s of Type 2 from U_2. So $|U_2^*| \leq |U_2 \cup U_0| + |X_1| + 1 \leq |U_2 \cup U_0| + s$ and $|V_2^*| \leq |V_2 \cup V_0| + t - p < |V_2 \cup V_0| + t$. Then by (13) and (14) we have $|U_2^*| < (k+1)(s+t)$ and $|V_2^*| < k(s+t) + 2t$, contradicting Claim 3.5.

Case 1.2.ii. $1 \leq |X_2|$. By Claim 3.4, since there exists a crossing $K_{s,t}$ of Type 2 from U_2, then there can be no crossing $K_{s,t}$'s of Type 2 from U_1. So $|U_1^*| \leq |U_1 \cup U_0| + |X_2| < |U_1 \cup U_0| + s$ and $|V_1^*| \leq |V_1 \cup V_0| + p < |V_1 \cup V_0| + s$. Then by (13) we have $|U_1^*|, |V_1^*| < (k+1)(s+t)$, contradicting Claim 3.5.

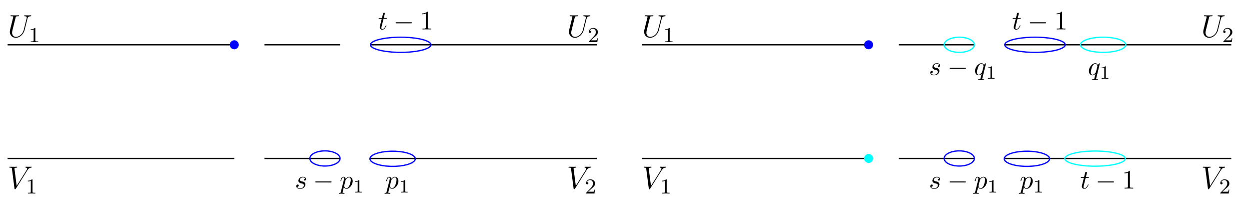


Figure 4: Case 2

Case 2. There are no crossing $K_{s,t}$'s of Type 1. By Claim 3.3, there can only be crossing $K_{s,t}$'s of Type 2. Without loss of generality suppose that $1 \leq |X_1|$. Then there can be no crossing $K_{s,t}$ of Type 2 from U_2 or V_2. So $|U_2^*| \leq |U_2 \cup U_0| + |X_1| < |U_2 \cup U_0| + s$ and $|V_2^*| \leq |V_2 \cup V_0| + |Y_1| < |V_2 \cup V_0| + s$. Then by (13) we have $|U_2^*|, |V_2^*| < (k+1)(s+t)$, contradicting Claim 3.5.

$\square$

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